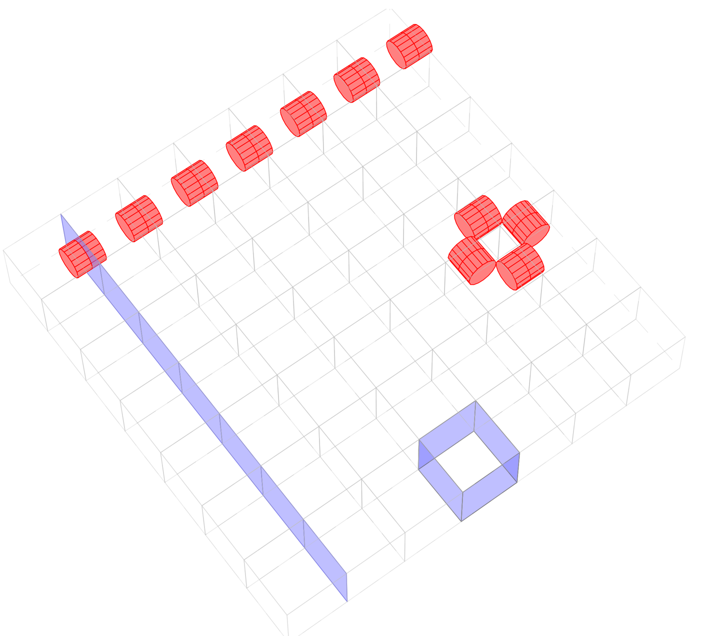
Higher-dimensional quantum hypergraph-product codes and concatenated codes

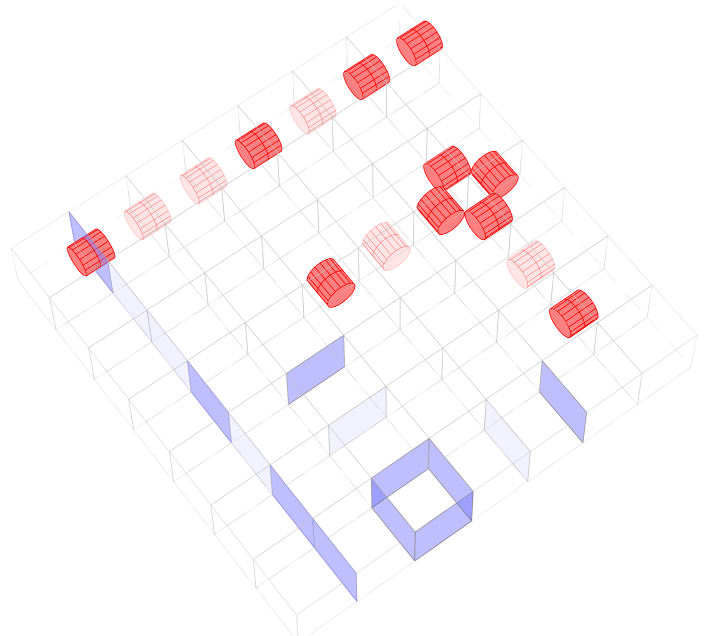
Weilei Zeng & Leonid P. Pryadko—University of California, Riverside

Abstract

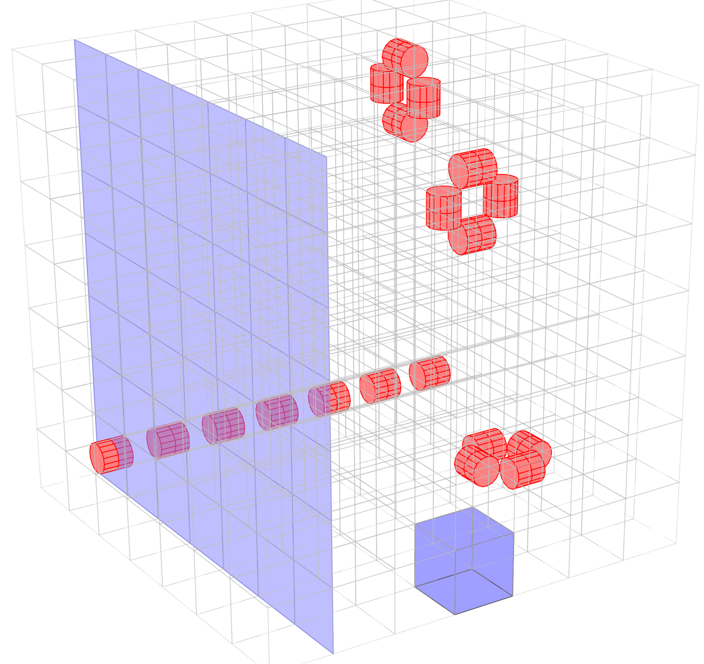
We describe a family of quantum error-correcting codes which generalize both the quantum hypergraph-product (QHP) codes by Tillich and Zémor, and all families of toric codes on m-dimensional hypercubic lattices. Similar to the former, these include finite-rate codes with distance scaling as a power of the block length. In addition to the proof in Ref.[1], we analyze the relation of the constructed codes to concatenated codes. Specifically, a map to concatenated codes can be use to give an alternative, much simpler proof of the lower bound on the distance. In more general case with chain complexes of arbitrary length, we show the distance is lower bounded by that of certain subsystem code, or, equivalently, a symmetric concatenated code. Based on extensive numerics, we conjecture that the corresponding distance matches the upper bound, which gives a stronger version of a conjecture in Ref. [1].



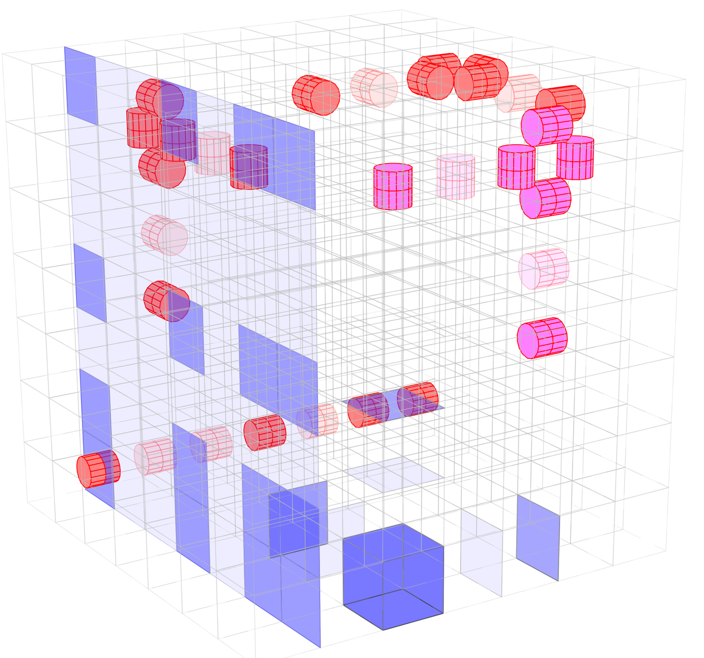
toric code $[[98,2,7]]$



QHP code $[[98,18,4]]$



3D toric code $[[1029,3,d_X=49,d_Z=7]]$



code $[[1029,81,d_X=16,d_Z=4]]$

Code construction

For any CSS code, the binary generator matrices H_X and H_Z must satisfy the orthogonality condition $H_X H_Z^T=0$. If we transpose the second matrix and define $A_1=H_X$, $A_2=H_Z^T$, the same condition reads $A_1A_2=0$. These two matrices can be used to define a 2-chain complex,

$$\mathcal{A}:\{0\}\stackrel{\partial}{\leftarrow}\mathcal{A}_0\stackrel{A_1}{\leftarrow}\mathcal{A}_1\stackrel{A_2}{\leftarrow}\mathcal{A}_2\stackrel{\partial}{\leftarrow}\{0\}$$

a sequence of three non-trivial spaces with two linear "boundary" maps between them given by A_1 and A_2. Orthogonality condition corresponds to the defining property "boundary of a boundary is zero".

The Z-codewords of the CSS code $[[n,k,d]]$ are elements of $\mathcal{A}_1=\mathbb{F}_2^n$, they are cycles, $A_1c^T=0$, which are not boundaries, $c\neq\alpha A_2^T$. They form the 1st homology group $\mathcal{H}_1(\mathcal{A})$ of rank k and distance $d_Z\geq d$.

Tensor product of chain complexes

$$\mathcal{A}:\{0\}\stackrel{\partial}{\leftarrow}\mathcal{A}_0\stackrel{A_1}{\leftarrow}\mathcal{A}_1\stackrel{A_2}{\leftarrow}\ldots\stackrel{A_a}{\leftarrow}\mathcal{A}_a\stackrel{\partial}{\leftarrow}\{0\},\ \mathcal{B}:\{0\}\stackrel{\partial}{\leftarrow}\mathcal{B}_0\stackrel{B_1}{\leftarrow}\mathcal{B}_1\stackrel{B_2}{\leftarrow}\ldots\stackrel{B_b}{\leftarrow}\mathcal{B}_b\stackrel{\partial}{\leftarrow}\{0\}.$$

Their tensor product $\mathcal{C}$ is a complex with spaces $\mathcal{C}_\ell=\bigoplus_{i+j=\ell}\mathcal{A}_i\otimes\mathcal{B}_j$ and the boundary operators $\partial(\boldsymbol{\alpha}\otimes\boldsymbol{\beta})=\partial_{\mathcal{A}}\boldsymbol{\alpha}\otimes\boldsymbol{\beta}\pm\boldsymbol{\alpha}\otimes\partial_{\mathcal{B}}\boldsymbol{\beta}$.

$$\begin{array}{ccccccc}\mathcal{C}_{-1} & & \mathcal{C}_0 & & \mathcal{C}_1 & & \mathcal{C}_2\ \ldots\\ \{0\} & \leftarrow & \mathcal{A}_0\otimes\mathcal{B}_0 & \leftarrow & \mathcal{A}_1\otimes\mathcal{B}_0 & \leftarrow & \mathcal{A}_2\otimes\mathcal{B}_0\ \ldots\\ & & & \nwarrow & & \nwarrow & \\ & & & & \mathcal{A}_0\otimes\mathcal{B}_1 & \leftarrow & \mathcal{A}_1\otimes\mathcal{B}_1\ \ldots\\ & & & & & \nwarrow & \\ & & & & & & \mathcal{A}_0\otimes\mathcal{B}_2\ \ldots\end{array}$$

$$C_1=\left(A_1\otimes E\,|\,E\otimes B_1\right),\quad C_2=\left(\begin{array}{c|c|c}A_2\otimes E & E\otimes B_1 & \\ \hline & -A_1\otimes E & E\otimes B_2\end{array}\right),\quad C_3=\left(\begin{array}{c|c|c|c}A_3\otimes E & E\otimes B_1 & & \\ \hline & -A_2\otimes E & E\otimes B_2 & \\ \hline & & A_1\otimes E & E\otimes B_3\end{array}\right),\ldots$$

Theorem (Künneth) Homology group $\mathcal{H}_\ell(\mathcal{C})\cong\bigoplus_{i+j=\ell}\mathcal{H}_i(\mathcal{A})\otimes\mathcal{H}_j(\mathcal{B})$, with $\operatorname{rank}\mathcal{H}_\ell(\mathcal{C})\equiv k_\ell^{(\mathcal{C})}=\sum_{i+j=\ell}k_i^{(\mathcal{A})}k_j^{(\mathcal{B})}$. and $\dim\mathcal{C}_\ell=\sum_{i+j=\ell}a_ib_j$, where $\dim\mathcal{A}_j=a_j$ and $\dim\mathcal{B}_j=b_j$, $j=0,1,\ldots,$

Do we also know the homological distances?

- Theorem (Tillich & Zémor '09) For a product of two 1-complexes $\mathcal{A}:\{0\}\stackrel{\partial}{\leftarrow}\mathcal{A}_0\stackrel{A_1}{\leftarrow}\mathcal{A}_1\stackrel{\partial}{\leftarrow}\{0\}$ and $\mathcal{B}:\{0\}\stackrel{\partial}{\leftarrow}\mathcal{B}_0\stackrel{B_1}{\leftarrow}\mathcal{B}_1\stackrel{\partial}{\leftarrow}\{0\}$, $c_1=a_0b_1+a_1b_0$, $k_1^{\mathcal{C}}=k_0^{\mathcal{A}}k_1^{\mathcal{B}}+k_1^{\mathcal{A}}k_0^{\mathcal{B}}$, and the distance $d_1^{\mathcal{C}}=\min\left(k_0^{\mathcal{A}}>0\,?\,d_1^{\mathcal{B}}:\infty,k_0^{\mathcal{B}}>0\,?\,d_1^{\mathcal{A}}:\infty\right)$. Convention used: $d_j=\infty$ whenever $k_j=0$.
- For products of 2-complexes, weaker lower bounds constructed in Ref. [3] and Ref. [4]
- Theorem (this work) Product of an m-complex $\mathcal{A}$ and a 1-complex $\mathcal{B}$,
  $$d_\ell^{\mathcal{C}}=\min\left(d_{\ell-1}^{\mathcal{A}}d_1^{\mathcal{B}},d_\ell^{\mathcal{A}}d_0^{\mathcal{B}}\right)$$ (same convention & notice $d_0\in\{1,\infty\}$).
- Conjecture (seen numerically): for a product of two complexes, $d_\ell^{\mathcal{C}}=\min_{i+j=\ell}d_i^{\mathcal{A}}d_j^{\mathcal{B}}$.

Lower distance bound via a subsystem code

Focus on the same (i,j) subspace of level $\ell=i+j$ in the product complex. That is, for any valid vector $z_{i+j}^{\mathcal{C}}$, drop any components outside of the subspace (i,j). This gives a CSS *subsystem* code with gauge generators

$$G_X=\begin{pmatrix}E\otimes B_j\\ A_i\otimes E\end{pmatrix},\quad G_Z^T=(E\otimes B_{j+1}|A_{i+1}\otimes E),\quad G_Z=\begin{pmatrix}E\otimes B_{j+1}^T\\ A_{i+1}^T\otimes E\end{pmatrix}.$$

The corresponding X stabilizer generator can be obtained as a linear combination of rows of G_X orthogonal to those of G_Z; we get

$$H_X=\begin{pmatrix}A_i\otimes B_j\\ X_i^{\mathcal{A}}\otimes B_j\\ A_i\otimes X_j^{\mathcal{B}}\end{pmatrix},\quad\text{where}\quad X_i^{\mathcal{A}}A_{i+1}=0,\quad X_j^{\mathcal{B}}B_{j+1}=0.$$

Here, rows of $X_i^{\mathcal{A}}$ form a basis of non-trivial vectors in the i th co-homology group. Alternatively, these are x-codewords in the CSS code with stabilizer generators $H_X=A_i$, $H_Z=A_{i+1}^T$; we can choose the canonical condition $Z_i^{\mathcal{A}}(X_i^{\mathcal{A}})^T=E$.

Consider the case where $\mathcal{B}:\ \{0\}\stackrel{\partial}{\leftarrow}\mathcal{B}_0\stackrel{B_1}{\leftarrow}\mathcal{B}_1\stackrel{\partial}{\leftarrow}\{0\}$ is a 1-complex. Here $j=1$, B_2 is empty, and the only condition on rows of $X_1^{\mathcal{B}}$ is that they be linearly independent from those of B_1. In this case we have en equivalent form,

$$H_X=\begin{pmatrix}A_i\otimes B_j\\ X_i^{\mathcal{A}}\otimes B_j\\ A_i\otimes X_j^{\mathcal{B}}\end{pmatrix}\Leftrightarrow\begin{pmatrix}A_i\otimes E\\ X_i^{\mathcal{A}}\otimes\mathcal{B}_1\end{pmatrix},\quad G_Z=(A_{i+1}^T\otimes E),$$

which is a special case of the concatenated CSS code. The distance of concatenated code is well known, $d_Z^{\text{concat}}=d_Z^{\mathcal{A}}d_Z^{\mathcal{B}}$, which gives a simple proof of the Theorem.

Concatenation: codewords of several copies of the inner code $\mathcal{A}$ used as qubits in the outer code $\mathcal{B}$. The code has following stabilizer generators and distance $d_Z^{\text{concat}}=d_Z^{\mathcal{A}}d_Z^{\mathcal{B}}$, $d_X^{\text{concat}}=d_X^{\mathcal{A}}d_X^{\mathcal{B}}$.

$$H_X=\begin{pmatrix}A_i\otimes E\\ X_i^{\mathcal{A}}\otimes B_j\end{pmatrix},\quad H_Z=\begin{pmatrix}A_{i+1}^T\otimes E\\ Z_i^{\mathcal{A}}\otimes B_{j+1}^T\end{pmatrix};$$

Symmetric concatenation from product of chain complexes:

$$H_X=\begin{pmatrix}A_i\otimes B_j\\ A_i\otimes X_j^{\mathcal{B}}\\ X_i^{\mathcal{A}}\otimes B_j\end{pmatrix},\quad G_Z=\begin{pmatrix}A_{i+1}^T\otimes E\\ E\otimes B_{j+1}^T\end{pmatrix}.$$

Conjecture (stronger version, seen numerically): $d_Z^{\text{symm. concat}}=d_Z^{\text{concat}}$.

Examples

- (TZ '09): QHP codes are product of two 1-complexes
- m-fold product of $\mathcal{A}$ generated by an $a_1\times a_1$ circulant matrix A_1, check matrix of a circulant code $[a_1,\kappa_1,\delta_1]$. We have
  $$n_j^{(m)}=\binom{m}{j}a_1^m,\quad k_j^{(m)}=\binom{m}{j}\kappa_1^m,\quad d_j^{(m)}=\delta_1^j.$$
- If A_1 is generated by $a(x)=1+x$ (repetition code $[a_1,1,a_1]$), we get the m-dimensional toric codes.

Potential applications

- Quantum LDPC codes with bounded-weight generators are FT whenever distance grows with the block length logarithmically or faster. Very few constructions of finite-R q-LDPC codes with provable (logarithmic or larger) lower bound on the distance exist. Even fewer have exact parameters known: QHP codes was the only example thus far.
  Our construction extends this class substantially.
- We expect that, similar to the toric codes, a number of related *finite-rate* code families can be constructed, e.g., analogs of rotated toric codes in 2D, analogs of color codes, and analogs of subsystem color codes.
- 4D toric codes have large redundancy in low-weight stabilizer generators. Such a redundancy can be used to improve accuracy of generator measurements, even single-shot QEC. Present construction allows single-shot fault-tolerant QEC with finite-R codes [4]
- Also: transformations between different QECCs, like the distance-balancing trick by Hastings, and asymmetric quantum CSS codes optimized for operation where error rates for X and Z channels may differ strongly.

References

[1] W. Zeng and L. P. Pryadko, Higher-dimensional quantum hypergraph-product codes, *Phys. Rev. Lett.* 122, 230501 (2019)

[2] J.-P. Tillich and G. Zemor, Quantum LDPC codes with positive rate and minimum distance proportional to $\sqrt{n}$, in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, pp. 799–803, (2009).

[3] B. Audoux and A. Couvreur, On tensor products of CSS codes, to appear in Ann. Inst. Henri Poincaré D (2019), **1512.07081**.

[4] E. T. Campbell, A theory of single-shot error correction for adversarial noise, *Quantum Science and Technology* (2019)